EAS 8803 – Environmental Data Analysis – Practice Exam

Answers provided by Jelinek and Ning (Pages 1/21)

1) Given two independent variables x and y with a Gaussian PDF, define a new set of variables
$$\hat{x} = a_{11}x + a_{12}y$$
$$\hat{y} = a_{21}x + a_{22}y$$

a) What is the covariance between $\hat{x}$ and $\hat{y}$?

$$<\hat{x}\hat{y}> = <(a_{11}x + a_{12}y)(a_{21}x + a_{22}y)>$$
$$<\hat{x}\hat{y}> = a_{11}a_{21} <xx> + a_{12}a_{22} <yy>$$

b) What is the joint PDF $P_{\hat{x}\hat{y}}$?

$$P_{\hat{x}\hat{y}} \frac{\partial(\hat{x}\hat{y})}{\partial(xy)} = P_{xy}$$
$$\frac{\partial(\hat{x}\hat{y})}{\partial(xy)} = \frac{\partial \hat{x}}{\partial x}\frac{\partial \hat{y}}{\partial y} - \frac{\partial \hat{x}}{\partial y}\frac{\partial \hat{y}}{\partial x} = a_{11}a_{22} - a_{12}a_{21}$$
$$P_{\hat{x}\hat{y}} = \frac{P_{xy}}{(a_{11}a_{22} - a_{12}a_{21})}$$
$$P_{\hat{x}\hat{y}} = \frac{P_x P_y}{(a_{11}a_{22} - a_{12}a_{21})}$$
$$P_{\hat{x}\hat{y}} = \frac{1}{2\pi\sigma_x\sigma_y}\frac{1}{(a_{11}a_{22} - a_{12}a_{21})}\exp\left[-\frac{x^2 + y^2}{4\sigma_x^{\ 2}\sigma_y^{\ 2}}\right]$$

$$\hat{a}_{11} = f_1(a_{11}, a_{12}, a_{21}, a_{22})$$
$$\hat{a}_{12} = f_2(a_{11}, a_{12}, a_{21}, a_{22})$$
$$\hat{a}_{13} = f_3(a_{11}, a_{12}, a_{21}, a_{22})$$
$$\hat{a}_{14} = f_4(a_{11}, a_{12}, a_{21}, a_{22})$$
$$x = \hat{a}_{11}\hat{x} + \hat{a}_{12}\hat{y}$$
$$y = \hat{a}_{21}\hat{x} + \hat{a}_{22}\hat{y}$$

$$P_{\hat{x}\hat{y}} = \frac{1}{2\pi\sigma_x\sigma_y}\frac{1}{(a_{11}a_{22} - a_{12}a_{21})}\exp\left[-\frac{(\hat{a}_{11}\hat{x} + \hat{a}_{12}\hat{y})^2 + (\hat{a}_{21}\hat{x} + \hat{a}_{22}\hat{y})^2}{4\sigma_x^{\ 2}\sigma_y^{\ 2}}\right]$$
$$P_{\hat{x}\hat{y}} = \frac{1}{2\pi\sigma_x\sigma_y}\frac{1}{(a_{11}a_{22} - a_{12}a_{21})}\exp\left[-\frac{(\hat{a}_{11}^{\ 2}\hat{a}_{21}^{\ 2})\hat{x}^2 + 2(\hat{a}_{11}\hat{a}_{12} + \hat{a}_{21}\hat{a}_{22})\hat{x}\hat{y} + (\hat{a}_{12}^{\ 2}\hat{a}_{22}^{\ 2})\hat{y}^2}{4\sigma_x^{\ 2}\sigma_y^{\ 2}}\right]$$

cross correlation term

2) Given a model $y = \alpha x + n$

a) Compute α using the statistics of x and y

$$\alpha = \frac{<xy>}{<xx>}$$

b) Show that the estimate of α is equivalent to minimizing the error variance using LSQ

$$J = <n^2> = <(y - \alpha x)^2>$$
$$J = <y^2 - 2\alpha xy + \alpha^2 x^2>$$
$$J = <y^2> - 2\alpha <xy> + \alpha^2 <x^2>$$
$$\frac{\partial J}{\partial \alpha} = -2 <xy> + 2\alpha <x^2> = 0$$
$$\Rightarrow \alpha = \frac{<xy>}{<xx>}$$

So the result is the same

c) Show that the square of the correlation coefficient $\rho = \dfrac{<xy>}{\sqrt{<x^2><y^2>}}$ is the fraction of variance of $<y^2>$ that is explained by x

$$\text{goal} \;\Rightarrow\; <y^2>(1-\rho^2) = <n^2>$$
$$\rho^2 = \frac{<xy>^2}{<x^2><y^2>}$$
$$<y^2>\left(1 - \frac{<xy>^2}{<x^2><y^2>}\right) = <y^2> - \frac{<xy>^2}{<xx>} = <y^2> - \alpha <xy>$$

As $y = \alpha x + n$

$$<(\alpha x + n)> - \alpha <x(\alpha x + n)>$$
$$\alpha^2 <x^2> + <n^2> - \alpha <x(\alpha x + n)>$$
$$<n^2> = \alpha^2 <x^2> + <n^2> - \alpha^2 <x^2>$$
$$<n^2> = <y^2>(1-\rho^2)$$

3) Given the two dependent variables x and y, define a set $\begin{aligned}\hat{x} &= x\cos\phi + y\sin\phi \\ \hat{y} &= -x\sin\phi + y\cos\phi\end{aligned}$

a) Find the angle ϕ for which $\hat{x}$ and $\hat{y}$ are uncorrelated?

This is solved via the covariance equal to zero:

$$< \hat{x}\hat{y} > = < (x\cos\phi + y\sin\phi)(-x\sin\phi + y\cos\phi) >$$

$$0 = < (x\cos\phi + y\sin\phi)(-x\sin\phi + y\cos\phi) >$$

$$0 = < -x^2\cos\phi\sin\phi - xy\sin^2\phi + yx\cos^2\phi + y^2\sin\phi\cos\phi >$$

$$0 = < (y^2 - x^2)\cos\phi\sin\phi - xy(\sin^2\phi - \cos^2\phi) >$$

$$0 = \frac{1}{2}\sin 2\phi(< y^2 > - < x^2 >) + < xy > \cos 2\phi$$

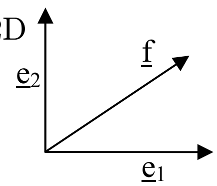
$$\tan 2\phi = \frac{2 < xy >}{< x^2 > - < y^2 >}$$

$$\phi = \frac{1}{2}\arctan\frac{2 < xy >}{< x^2 > - < y^2 >}$$

b) Can you apply this rotation to any two time series to render them uncorrelated?

yes

4) Consider the Cartesian basis in 2D



$$\underline{e}_1 = \begin{bmatrix}1\\0\end{bmatrix} \quad \underline{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$$

Any vector f can be expressed according to the projection on the two axis defined by $\underline{e}_1$ and $\underline{e}_2$. $\underline{f} = \underline{e}_1\alpha_1 + \underline{e}_2\alpha_2$ or more generally $\underline{f} = \sum_{i=1}^{N} \alpha_i \underline{e}_i$. The set $\underline{e}_i$ is said to be orthogonal if $\underline{e}_i$ has the property that no $\underline{e}_i$ can be represented by using another $\underline{e}_j$. e.g. $\underline{e}_1\alpha_1 \neq \underline{e}_2$ because $\underline{e}_1$ is orthogonal to $\underline{e}_2$, more generally $\underline{e}_i^T \underline{e}_j = \delta_{ij}$

a) Show that $\underline{e}_1$ and $\underline{e}_2$ are indeed orthogonal by direct multiplication.

$$\underline{e}_1^T \underline{e}_2 = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = 1\times 0 + 0\times 1 = 0 \quad \text{So they are indeed orthogonal}$$

b) How can you use the orthogonality to estimate α_1 projection for the vector $\underline{f}$?

$$\underline{e}_1^T f = \underline{e}_1^T(\alpha_1 \underline{e}_1 + \alpha_2 \underline{e}_2)$$

$$\underline{e}_1^T f = \alpha_1 \underline{e}_1^T \underline{e}_1 + \alpha_2 \underbrace{\underline{e}_1^T \underline{e}_2}_{0}$$

$$\underline{e}_1^T f = \alpha_1 \underbrace{\underline{e}_1^T \underline{e}_1}_{1}$$

$$\alpha = \underline{e}_1^T f$$

5) Assume that f(t) is a continuous function given by the following Fourier Series $f(t) = \sum_{n=1}^{N} [A_n \cos(2\pi s_n t) + B_n \sin(2\pi s_n t)]$

a) What are the projections of f(t) on the basis $\begin{matrix} \underline{e}_1(t) = \cos(s_{10} 2\pi t) \\ \underline{e}_2(t) = \sin(s_{10} 2\pi t) \end{matrix}$?

$$\Delta t e_1^T f = \int_{-\infty}^{\infty} e_1(t_{10}) f(t) dt \qquad t = (0, T) \quad s_{10} = 10/T$$

$$\Delta t e_1^T f = \int_0^T \cos(2\pi s_{10} t) A_{10} \cos(2\pi s_{10} t) dt$$

$$\Delta t e_1^T f = A_{10} \left(\frac{t}{2} + \frac{\sin(4\pi s_{10} t)}{8\pi s_{10}} \right) \Bigg|_0^T$$

$$\Delta t e_2^T f = \int_{-\infty}^{\infty} e_2(t_{10}) f(t) dt \qquad t = (0, T) \quad s_{10} = 10/T$$

$$\Delta t e_1^T f = \int_0^T \sin(2\pi s_{10} t) B_{10} \sin(2\pi s_{10} t) dt$$

$$\Delta t e_1^T f = B_{10} \left(\frac{t}{2} - \frac{\sin(4\pi s_{10} t)}{8\pi s_{10}} \right) \Bigg|_0^T$$

b) Are $\underline{e}_1$ and $\underline{e}_2$ orthogonal?

$$\int_{-\infty}^{\infty} \underline{e}_1(t) \underline{e}_2(t) dt = \int_{-\infty}^{\infty} \cos(s_{10} 2\pi t) \sin(s_{10} 2\pi t)$$

$$= \frac{1}{4\pi s_{10}} \sin^2(s_{10} 2\pi t) \Bigg|_{-\infty}^{\infty}$$

As $\sin(s_{10} 2\pi t)$ is an even function of t, this means $\sin^2(s_{10} 2\pi t) = \sin^2(-s_{10} 2\pi t)$ then the integration would = 0, so yes they are orthogonal.

6) Assume $\underline{y}=\underline{\underline{A}}\underline{m}+\underline{n}$ where $<\underline{n}\underline{n}^T>=\sigma_n^2\underline{\underline{I}}$. $\underline{\underline{A}}$ is a known MxN matrix and $\underline{y}$ a measured quantity.

a) When is the system $\underline{y}=\underline{\underline{A}}\underline{m}+\underline{n}$ overdetermined and underdetermined?

Overdetermined when M>N (more equations than parameters) and underdetermined when N<M (more parameters than equations)

b) Find $\hat{\underline{m}}$ using LSQ?

$$J=\underline{n}^T\underline{n}=(\underline{y}-\underline{\underline{A}}\underline{m})^T(\underline{y}-\underline{\underline{A}}\underline{m})$$

$$\frac{\partial J}{\partial \underline{m}}=2\left(\underline{\underline{A}}^T\underline{\underline{A}}\underline{m}-\underline{\underline{A}}^T\underline{y}\right)=0$$

$$\underline{\underline{A}}^T\underline{\underline{A}}\underline{m}=\underline{\underline{A}}^T\underline{y}$$

$$\hat{\underline{m}}=\left(\underline{\underline{A}}^T\underline{\underline{A}}\right)^{-1}\underline{\underline{A}}^T\underline{y}$$

c) Now assume I told you $<\underline{m}\underline{m}^T>=\underline{\underline{P}}=\begin{bmatrix}\sigma_m^2 & & 0\\ & \ddots & \\ 0 & & \sigma_m^2\end{bmatrix}$, Find $\hat{\underline{m}}$ using this information.

$$J=\underline{n}^T\underline{n}+\underline{m}^T\underline{\underline{P}}^{-1}\underline{m}$$

$$\frac{\partial J}{\partial \underline{m}}=2\left(\underline{\underline{A}}^T\underline{\underline{A}}\underline{m}-\underline{\underline{A}}^T\underline{y}\right)+2\underline{\underline{P}}^{-1}\underline{m}=0$$

$$\left(\underline{\underline{A}}^T\underline{\underline{A}}+\underline{\underline{P}}^{-1}\right)\underline{m}=\underline{\underline{A}}^T\underline{y}$$

$$\hat{\underline{m}}=\left(\underline{\underline{A}}^T\underline{\underline{A}}+\underline{\underline{P}}^{-1}\right)^{-1}\underline{\underline{A}}^T\underline{y}$$

d) Estimate the uncertainty on your estimate of $\hat{\underline{m}}$ by computing $\hat{\underline{\underline{P}}}=<(\underline{m}-\hat{\underline{m}})(\underline{m}-\hat{\underline{m}})^T>$. Is this matrix diagonal and equal to $\underline{\underline{P}}$?

$$\hat{\underline{\underline{P}}}=<(\underline{m}-\hat{\underline{m}})(\underline{m}-\hat{\underline{m}})^T>$$

$$\hat{\underline{\underline{P}}}=<(\underline{\underline{A}}^T\underline{\underline{A}}+\underline{\underline{P}}^{-1})^{-1}\underline{\underline{A}}^T\underline{n}\left[(\underline{\underline{A}}^T\underline{\underline{A}}+\underline{\underline{P}}^{-1})^{-1}\underline{\underline{A}}^T\underline{n}\right]^T>$$

$$\hat{\underline{\underline{P}}}=<(\underline{\underline{A}}^T\underline{\underline{A}}+\underline{\underline{P}}^{-1})^{-1}\underline{\underline{A}}^T\underline{n}\underline{n}^T\underline{\underline{A}}\left[(\underline{\underline{A}}^T\underline{\underline{A}}+\underline{\underline{P}}^{-1})^{-1}\right]^T>$$

$$\hat{\underline{\underline{P}}}=(\underline{\underline{A}}^T\underline{\underline{A}}+\underline{\underline{P}}^{-1})^{-1}\underline{\underline{A}}^T\left\langle\underline{n}\underline{n}^T\right\rangle\underline{\underline{A}}\left[(\underline{\underline{A}}^T\underline{\underline{A}}+\underline{\underline{P}}^{-1})^{-1}\right]^T$$

$$\hat{\underline{\underline{P}}}=(\underline{\underline{A}}^T\underline{\underline{A}}+\underline{\underline{P}}^{-1})^{-1}\underline{\underline{A}}^T\sigma_n^2\underline{\underline{I}}\underline{\underline{A}}\left[(\underline{\underline{A}}^T\underline{\underline{A}}+\underline{\underline{P}}^{-1})^{-1}\right]^T$$

The matrix of the uncertainty on your estimate of the model parameters is not necessarily diagonal and depends only on the model and the statistics of the measurement errors $\sigma_n^2 \underline{\underline{I}}$.

7) Now assume $\underline{y} = \underline{\underline{A}}\,\underline{m} + \underline{n}$ like in #6. Suppose I also told you that $\underline{\underline{B}}\,\underline{m} = \underline{x}$ exactly.

a) Write down a cost function to include this information, Include also $\underline{\underline{P}}$.

$$J = \underline{n}^T \underline{n} + \underline{m}^T \underline{\underline{P}}^{-1} \underline{m} + \underline{\mu}^T (\underline{\underline{B}}\,\underline{m} - \underline{x})$$

$$J = \left(\underline{y} - \underline{\underline{A}}\,\underline{m}\right)^T \left(\underline{y} - \underline{\underline{A}}\,\underline{m}\right) + \underline{m}^T \underline{\underline{P}}^{-1} \underline{m} + \underline{\mu}^T (\underline{\underline{B}}\,\underline{m} - \underline{x})$$

b) Derive the equations for an extreme of your cost function. (These are also called the normal equations)

$$\frac{\partial J}{\partial \underline{\mu}} = 0 \Rightarrow (\underline{\underline{B}}\,\underline{m} - \underline{x}) = 0$$

$$\frac{\partial J}{\partial \underline{m}} = 0 \Rightarrow 2\left(\underline{\underline{A}}^T \underline{\underline{A}}\,\underline{m} - \underline{\underline{A}}^T \underline{y}\right) + 2\underline{\underline{P}}^{-1} \underline{m} + \underline{\mu}^T \underline{\underline{B}} = 0$$

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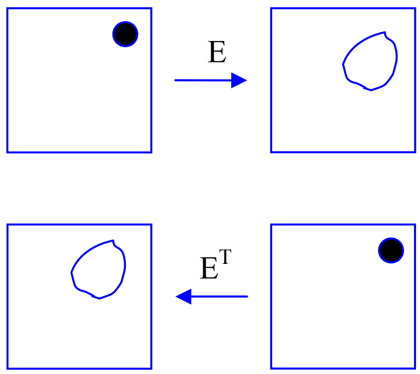
Answers provided by Jelinek and Ning (Pages 9/21)

8) Given $\underline{y} = \underline{\underline{E}}\,\underline{x} + \underline{n}$, $\underline{\underline{E}}^T$ is said to be the adjoint of $\underline{\underline{E}}$, your model.

a) What is the meaning of the adjoint $\underline{\underline{E}}^T$? (In words)



Where in a normal situation E tells you about the correlation behavior between your input point at t=0 and a correlated area at t=N via the covariance of input via output, the adjoint model or ET tells you about the output point at T=N and a correlated area at t=0 via the covariance of output times input

b) Based on the statistics of $\underline{x}$ and $\underline{y}$ show that $\underline{\underline{E}} = \langle \underline{y}\,\underline{x}^T \rangle \langle \underline{x}\,\underline{x}^T \rangle^{-1}$

$$\underline{y} = \underline{\underline{E}}\,\underline{x} + \underline{n}$$

$$\underline{y}\,\underline{x}^T = \underline{\underline{E}}\,\underline{x}\,\underline{x}^T + \underline{n}\,\underline{x}^T$$

$$\left\langle \underline{y}\,\underline{x}^T \right\rangle = \underline{\underline{E}} \left\langle \underline{x}\,\underline{x}^T \right\rangle + \underbrace{\left\langle \underline{n}\,\underline{x}^T \right\rangle}_{0}$$

$$\left\langle \underline{y}\,\underline{x}^T \right\rangle \left\langle \underline{x}\,\underline{x}^T \right\rangle^{-1} = \underline{\underline{E}} \underbrace{\left\langle \underline{x}\,\underline{x}^T \right\rangle \left\langle \underline{x}\,\underline{x}^T \right\rangle^{-1}}_{Identity\ Matrix}$$

$$\left\langle \underline{y}\,\underline{x}^T \right\rangle \left\langle \underline{x}\,\underline{x}^T \right\rangle^{-1} = \underline{\underline{E}}$$

c) Compute the Adjoint $\underline{\underline{E}}^T$ and discuss its meaning based on what you find. Does it match what you said in a).

$$\underline{\underline{E}}^T = \left(\left\langle \underline{y}\,\underline{x}^T \right\rangle \left\langle \underline{x}\,\underline{x}^T \right\rangle^{-1} \right)^T$$

$$\underline{\underline{E}}^T = \left\langle \underline{x}\,\underline{x}^T \right\rangle^{-1} \left\langle \underline{y}\,\underline{x}^T \right\rangle^T$$

$$\underline{\underline{E}}^T = \left\langle \underline{x}\,\underline{x}^T \right\rangle^{-1} \left\langle \underline{x}\,\underline{y}^T \right\rangle$$

The adjoint matrix is now the backward covariance between the outputs and the input. In other words it tells us how one desired output $\underline{y}$ is influenced by the input parameters $\underline{x}$. In the case of $\underline{\underline{E}} = \langle \underline{y}\,\underline{x}^T \rangle \langle \underline{x}\,\underline{x}^T \rangle^{-1}$ we get the reverse, in that we have information of how one input parameters affects the outputs.

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8) Given a series y(t) with stationary statistics.

a) Define what it means to be stationary.

A series with stationary statistics tells us that the true mean of the variable and its higher-order moments are independent from the time in question. This typically requires the statistics to be detrended before they can be used.

b) Derive the autocovariance function $\alpha(\tau)$ where □ is the time lag.

$$y(t-\tau) = \alpha y(t) + n(t)$$

$$\alpha = < y(t-\tau) y(t) >$$ autocovariance

c) Show that the Fourier Transform of $\alpha(\tau)$ is equal to the spectrum $\hat{y}(s)\hat{y}^*(s)$

$$\alpha(\tau) = \int_{-\infty}^{\infty} y(t-\tau) y(t) dt$$

$$\alpha(\tau) = \int_{-\infty}^{\infty} y(\tau) \int_{-\infty}^{\infty} \hat{y}(s) e^{i2\pi s(t-\tau)} ds dt'$$

$$\alpha(\tau) = \int_{-\infty}^{\infty} y(\tau) \int_{-\infty}^{\infty} \hat{y}^*(\tau) e^{i2\pi s\tau} dt' e^{i2\pi st} ds \qquad \hat{y}^*(s)$$

$$\alpha(\tau) = \int_{-\infty}^{\infty} y(s) \hat{y}^*(s) e^{i2\pi s\tau} ds \qquad \hat{h}(s)$$

9) Given the following equations

1. $\frac{dy}{dt} = -2\pi s_b y + f(t) \qquad f(t)$ is white noise forcing

2. $\frac{dy}{dt} = -2\pi s_b y + A\cos(2\pi s_a t) + f(t)$

with $S_b < S_a$

a) Compute the analytical spectrum of y(t) in 1 and 2

$$1. \frac{dy}{dt} = -2\pi s_b y + f(t)$$

$$y(t) = \int_{-\infty}^{\infty} y(s) e^{-i2\pi st} ds$$

$$\int_{-\infty}^{\infty} y(s) e^{-i2\pi st}(-2\pi is) ds = -2\pi s_b \int_{-\infty}^{\infty} y(s) e^{-i2\pi st} ds + f(t)$$

$$f(t) = \int_{-\infty}^{\infty} f(s) e^{-i2\pi st} ds$$

$$\Rightarrow \int_{-\infty}^{\infty} y(s) e^{-i2\pi st}(2\pi s_b - 2\pi is) ds = \int_{-\infty}^{\infty} f(s) e^{-i2\pi st} ds$$

$$\Rightarrow f(s) = y(s)(2\pi s_b - 2\pi is)$$

$$\Rightarrow f(s) f^*(s) = y(s) y^*(s)(2\pi s_b - 2\pi is)(2\pi s_b + 2\pi is)$$

$$\Rightarrow f(s) f^*(s) = y(s) y^*(s)(4\pi^2 s_b^{\ 2} + 4\pi^2 s^2)$$

$$\Rightarrow y(s) y^*(s) = \frac{1}{4\pi^2} \frac{f(s) f^*(s)}{s_b^{\ 2} + s^2}$$

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$$2.\frac{dy}{dt} = -2\pi s_b y + A\cos(2\pi s_a t) + f(t)$$

$$y(t) = \int_{-\infty}^{\infty} y(s)e^{-i2\pi st}ds$$

$$\int_{-\infty}^{\infty} y(s)e^{-i2\pi st}(-2\pi is)ds = -2\pi s_b \int_{-\infty}^{\infty} y(s)e^{-i2\pi st}ds + A\cos(2\pi s_a t) + f(t)$$

$$f(t) = \int_{-\infty}^{\infty} f(s)e^{-i2\pi st}ds$$

$$A\cos(2\pi s_a t) = \int_{-\infty}^{\infty} \frac{1}{2}A(\delta(s-s_a)+\delta(s+s_a))e^{-i2\pi st}ds$$

$$\Rightarrow y(s)(-2\pi is) = -2\pi s_b y(s) + \frac{1}{2}A(\delta(s-s_a)+\delta(s+s_a)) + f(s)$$

$$\Rightarrow y(s)(2\pi s_b - 2\pi is) = \frac{1}{2}A(\delta(s-s_a)+\delta(s+s_a)) + f(s)$$

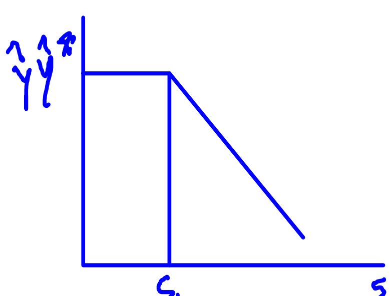
$$\Rightarrow y(s)y^*(s)(4\pi^2 s_b^{\,2} + 4\pi^2 s^2) = \left[\frac{1}{2}A(\delta(s-s_a)+\delta(s+s_a)) + f(s)\right]\left[\frac{1}{2}A(\delta(s-s_a)+\delta(s+s_a)) + f^*(s)\right]$$

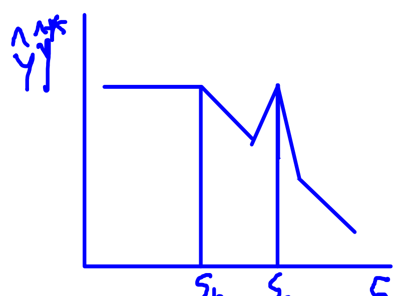
$$\Rightarrow y(s)y^*(s)(4\pi^2 s_b^{\,2} + 4\pi^2 s^2) = \frac{1}{4}A^2(\delta(s-s_a)+\delta(s+s_a))^2 + \frac{1}{2}\Big((f(s)+f^*(s))A(\delta(s-s_a)+\delta(s+s_a))\Big) + f(s)f^*(s)$$

$$\because \delta(s-s_a)\delta(s+s_a) = 0$$

$$\Rightarrow y(s)y^*(s) = \frac{1}{4\pi^2}\frac{\frac{1}{4}A^2(\delta^2(s-s_a)+\delta^2(s+s_a)) + f(s)f^*(s) + \frac{1}{2}A\Big((f(s_a)+f^*(s_a)+(f(-s_a)+f^*(-s_a)\Big)}{s_b^{\,2}+s^2}$$

b) Draw the spectra and label the frequency S_b, S_a





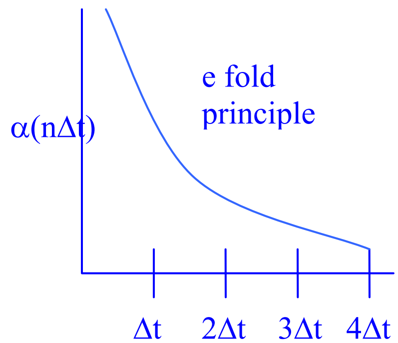
c) Discuss the shape of the spectra

The first spectra shape is consistent with a red noise signal until such time as s_b is obtained at which point the f term becomes dominant and white noise takes over.

The second spectra is quite similar to the first except for the additional spectra spike related to the dominance of the behavior of the delta function term at s_a. Since the remainder of the terms are the same the spectra is identical to the first at all other points.

10) Suppose you have a time series $y(t_n)$ $t_n = n\Delta T$ $n = 1 \ldots N$ with an autocorrelation $\alpha(n\Delta T) = e^{-n\Delta T / 4\Delta T}$

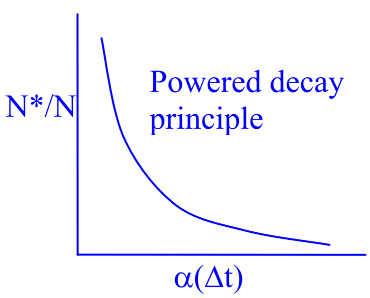
a) Plot $\alpha(n\Delta T)$ as a function of ΔT



b) What are the degrees of freedom of $y(t_n)$

Normally you would have $D.O.F.\, y(t_n) = N - 1$ however in this case we have a different N which is N* where $N^* = \frac{N\Delta t}{4\Delta t} = \frac{N}{4}$ so $D.O.F.\, y(t_n) = \frac{N}{4}$

c) Explain why the actual degrees of freedom N*¡N and make a plot of



Because each sample has a correlation with other samples they are no longer independent and therefore we can not have $D.O.F.\, y(t_n) = N - 1$ and must use our newly derived N* value. The reason we have N*¡N is because N is inclusive of the entire range of many realizations of N* making N*¡N always apply.

d) Use $\alpha(\Delta T)$ to write down the coefficients a_1 and a_2 of an AR-2 (autoregressive process of order 2) $x(t_{n+1}) = a_1 x(t_{n-1}) + a_2 x(t_{n-2}) + n(t_n)$

$$x(t_{n+1}) = a_1 x(t_n) + a_2 x(t_{n-1}) + n(t_n)$$

$$\langle x(t_{n+1}) x(t_n) \rangle = a_1 \langle x(t_n) x(t_n) \rangle + a_2 \underbrace{\langle x(t_{n-1}) x(t_n) \rangle}_{=\langle x(t_{n+1}) x(t_n) \rangle}$$

$$(1 - a_2) \langle x(t_{n+1}) x(t_n) \rangle = a_1 \langle x(t_n) x(t_n) \rangle$$

$$a_1 = (1 - a_2) \frac{\langle x(t_{n+1}) x(t_n) \rangle}{\langle x(t_n) x(t_n) \rangle} = (1 - a_2) \alpha(\Delta t)$$

$$a_2 = (1 - a_1) \frac{\langle x(t_{n+1}) x(t_{n-1}) \rangle}{\langle x(t_n) x(t_n) \rangle} = (1 - a_1) \alpha(2\Delta t)$$

11) Assume you have the following dynamics $\frac{d^2y}{dt^2} = -s_b y + f(t) + A\cos(2\pi s_a t)$ and you computed the spectrum of y(t). You see some peaks and you want to know if they are significant.

a) Which AR process would you use as your null hypothesis (NOTE: $\frac{d^2y}{dt^2} = \frac{y_{t+1} + y_{t-1} - 2y_t}{\Delta t^2}$)

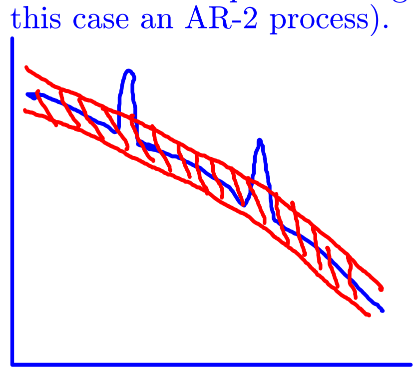
You would use the AR-2 process because we evaluate how many time steps we use in our process and in this case there are 2 autoregressive terms.

b) If the spectral shape of the AR process you picked is given by the function $P_{AR}(s)$ and you had 10 realizations of y(t) to use for computing $y(s)y^*(s)$, how would you estimate the 95%

First we note that with each spectral estimate we have 2 degrees of freedom. Because we have 10 estimates of the spectra, each with 2 d.o.f., this gives us a total of 20 d.o.f.. Now as we've been given $P_{AR}(s)$ (the variance at each frequency), we can estimate the confidence limits of the true variance σ^2, for each frequency using the chi-square distribution with 20 degrees of freedom (recall the test of variance, see question 15 or Chapter 1 of Hartman PDF notes)

$$\frac{20}{\chi^2_{97.5\%}} P_{AR}(s) < \sigma^2(s) < \frac{20}{\chi^2_{2.5\%}} P_{AR}(s)$$

Now we have the upper $\frac{20}{\chi^2_{2.5\%}} P_{AR}(s)$ and lower bound $\frac{20}{\chi^2_{97.5\%}} P_{AR}(s)$ for the variance at each frequency. These can be drawn on top of the spectra to evaluate which peaks are significant with respect to the null hypothesis (in this case an AR-2 process).



c) Without solving for the spectra how many significant peaks would you guess and at what frequency?

You would anticipate two peaks based on this equation. One would occur at the frequency of s_b , given that this is a wave equation and s_b is the frequency of the traveling waves. The other would occur at the frequency of s_a of the forcing.

12) The Discrete Fourier Transform

$$y(t_n) = A_0 + \sum_{k=1}^{N/2-1}\left\{A_k \cos\left(\frac{2\pi k t_n}{T}\right) + B_k \sin\left(\frac{2\pi k t_n}{T}\right)\right\} + A_{N/2}\cos\left(\frac{\pi N t_n}{T}\right) \text{ where } n = 1 \ldots N \text{ and}$$

$T = N\Delta t$

a) What is the Nyquist frequency and its amplitude?

The Nyquist frequency is the maximum frequency you can resolve based on the length of a series that allows full reconstruction of the signal of a given DFT, its magnitude $A_{N/2}$ is defined by $N/2$ in this case.

b) Which is the mean?

The mean is the portion of the DFT that sits outside the summation, so in this case A_0

c) What is the variance explained by the periodicity $s_a = \dfrac{10}{T}$?

The variance is explained by the relationship $\dfrac{a_k^2 + b_k^2}{2}$, so in this case $\dfrac{A_{10}^2 + B_{10}^2}{2}$

d) Using the orthogonality relationship how can you compute A_{10} and B_{10}? (Assume $A_0 = 0$)

$$\underline{e}_{10}^T \underline{y} = \sum_{n=1}^{N} \cos\left(\frac{2\pi 10 t_n}{N\Delta t}\right) y(t_n) = \sum_{n=1}^{N} \cos\left(\frac{2\pi 10 t_n}{N\Delta t}\right) \underbrace{A_{10}\cos\left(\frac{2\pi 10 t_n}{N\Delta t}\right)}_{y(t_n) \text{ becuase all other frequencies integrate to zero}} = \frac{N}{2}A_{10}$$

$$\longrightarrow A_{10} = \frac{2}{N}\sum_{n=1}^{N} \cos\left(\frac{2\pi 10 t_n}{N\Delta t}\right) y(t_n)$$

the same is true for B_{10}

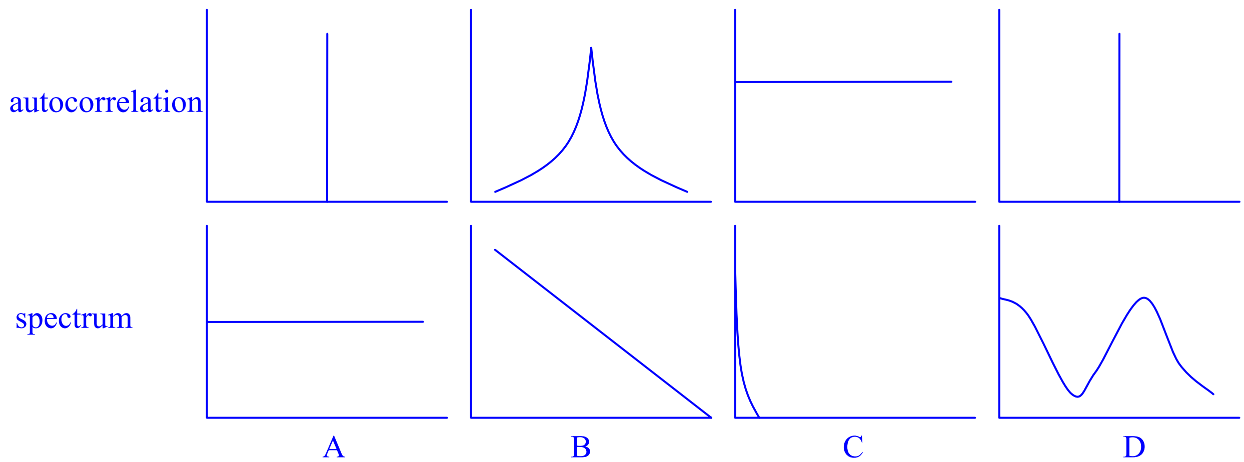
13) Draw the autocorrelation function and spectrum of the following processes

a) White noise

b) Red noise

c) y = at

d) $y = a\cos(2\pi S_a t)$



a) The autocorrelation of white noise is a single peak at 0, while its spectrum takes the general shape of a horizontal line.

b) The autocorrelation of the red noise is consistent with exponential decay centered on 0, while its spectrum takes the shape of a line slope that is defined by $1/s^2$

c) The autocorrelation is a horizontal line with the value at 1, while the spectrum peaks at a low frequency since its linear trend is bound by a single wave

d) The autocorrelation is a single peak and the spectrum takes the shape of a wave function

14) Tapering

a) Explain the concept of Tapering?

Tapering is a process that tapers the end of a signal towards zero at either end to reduce the variance contribution of the tails of the timeseries. This eliminates situations where you have abrupt starts and stops in the signal. This allows for a smoother start and stop. Tapering is particularly useful in dealing with issues of aliasing, leakage and discontinuities.

b) When we have a finite time series and we take a spectrum what kind of tapering is implicit? (If we do none) What does this do to the spectrum peaks estimate?

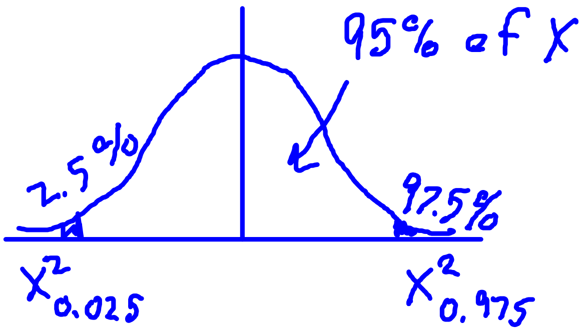
When taking a spectrum of a finite time series, a box car window is the implicit tapering method used. This will broaden the spectral peaks.

15) The test of variance says the confidence limit is $\frac{rs^2}{x^2_{0.025x}} < \sigma^2 < \frac{rs^2}{x^2_{0.975x}}$ where

$\sigma^2 =$ true variance

$s^2 =$ sample variance

where r = degree of freedom in the estimate of s^2 and



a) Given that the sample variance is computed from a variable $x(t_n)$ with $n=1...N$ derive the identity $rs^2 = \sigma^2 x^2$ where r are the degrees of freedom in computing s^2 r = N-1

A chi-square variable is defined as the sum

$$\chi^2 = \sum_{i=1}^{M} n_i^2$$

$$<n> = 0; \quad \sigma_n^2 = 1$$

This variable has a chi-square distribution with M degrees of freedom.

Now, the sample variance of $x(t_n)$ with $n=1...N$

$$s^2 = \frac{1}{N-1}\sum_{i=1}^{N} y_i^2 \longrightarrow (N-1)s^2 = \sum_{i=1}^{N} y_i^2$$

I also know that $\sum_{i=1}^{N} \frac{y_i^2}{\sigma_y^2}$ has a chi-square distribution, where σ_y^2 is the true unknown variance. If I have my chi-square variable defined as $\chi^2 = \sum_{i=1}^{N} \frac{y_i^2}{\sigma_y^2} = \frac{(N-1)s^2}{\sigma_y^2}$ this will have a chi-square distribution with N-1 degree of freedom. So then I can estimate the confidence limits of the variance by reading the value of $\chi^2{}_{2.5\%}$ and $\chi^2{}_{97.5\%}$ so that

$$\chi^2_{2.5\%} = \frac{(N-1)s^2}{\sigma^2_{y,2.5\%}} \longrightarrow \sigma^2_{y,2.5\%} = \frac{(N-1)s^2}{\chi^2_{2.5\%}} \quad \text{upper bound}$$

$$\chi^2_{97.5\%} = \frac{(N-1)s^2}{\sigma^2_{y,97.5\%}} \longrightarrow \sigma^2_{y,97.5\%} = \frac{(N-1)s^2}{\chi^2_{97.5\%}} \quad \text{lower bound}$$

b) Suppose $x(t_n)$ has an autocorrelation $\alpha(\Delta T) = e^{-\Delta T / 4\Delta T}$ and length N. How does that affect the confidence limit? Will they be higher or smaller?

For an equivalent timeseries of same length, the degrees of freedom would be N. If the timeseries is autocorrelated and the effective degrees of freedom are less (in this case N/4), this implies that you have less confidence in estimating the variance and your confidence limits must be higher! Do not get confuse by the formula where you find that the denominators will decrease with less d.o.f. , the $\chi^2_{97.5\%,d.o.f.}$ and $\chi^2_{2.5\%,d.o.f.}$ will change accordingly.

$$\frac{d.o.f.}{\chi^2_{97.5\%,d.o.f.}} s^2 < \sigma^2(s) < \frac{d.o.f.}{\chi^2_{2.5\%,d.o.f.}} s^2$$