Covariance Modeling in the TIME DOMAIN and Yule - Walker Equations

A generic form of a stochastic process in time can be expressed as a AR process of order N

$$y_t = a_1 y_{t-1} + a_2 y_{t-2} + \dots a_N y_{t-N} + \text{noise}$$

autoregressive terms

{Recall AR-1
If $a_n \ (n > 1) = \emptyset$ than $a_1 = r(\Delta t) = \dfrac{\langle y_t \, y_{t-1} \rangle}{\langle y_t \, y_t \rangle}$ }

In the general case we can rewrite the above time series using matrix notation

$$\underline{y} = \underline{\underline{\alpha}} \ \underline{x} + \underline{n}$$

$\underline{y} = [\, y_t \,]_{1\times 1}$ vector

$\underline{\underline{\alpha}} = [\, a_1 \ a_2 \ \dots \ a_N \,]$

$1 \times N$ matrix

$$\underline{x} = \begin{bmatrix} y_{t-1} \\ y_{t-2} \\ \vdots \\ y_{t-N} \end{bmatrix}$$ $N \times 1$ vector

we now derive the equations for $\underline{\underline{\alpha}}$

$$\langle \underline{y}\, \underline{x}^T \rangle = \underline{\underline{\alpha}} \langle \underline{x}\, \underline{x}^T \rangle$$ ← Normal Eq. solutions of LSQ.

$$\langle \underline{x}\, \underline{y}^T \rangle = \langle \underline{x}\, \underline{x}^T \rangle \underline{\underline{\alpha}}^T$$ ← These equations for $a_1, a_2 \dots a_N$ are also called Yule-Walker equations.

note that
$$\langle \underline{x}\, \underline{x}^T \rangle = \begin{bmatrix} \underbrace{\langle y(t-1)\, y(t-1) \rangle}_{R(0)} & \cdots & \underbrace{\langle y(t-1)\, y(t-N) \rangle}_{R(N)} \\ \vdots & \ddots & \vdots \\ \underbrace{\langle y(t-N)\, y(t-1) \rangle}_{R(N)} & \cdots & \underbrace{\langle y(t-N)\, y(t-N) \rangle}_{R(N)} \end{bmatrix}$$

if you define

$$\boxed{R(n) = \langle y(t)\, y(t-n) \rangle}$$ as the autocovariance function

than we have

$$\langle \underline{x}\, \underline{y}^T \rangle = \langle \underline{x}\, \underline{x}^T \rangle \underline{\underline{\alpha}}^T$$

Yule-Walker Eq. follow from cov. model.

$$\begin{bmatrix} R(1) \\ \vdots \\ \vdots \\ R(N+1) \end{bmatrix} = \begin{bmatrix} R(0) & R(1) & \cdots & R(N) \\ R(1) & R(0) & & \vdots \\ \vdots & & \ddots & \vdots \\ R(N) & \cdots & \cdots & R(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}$$

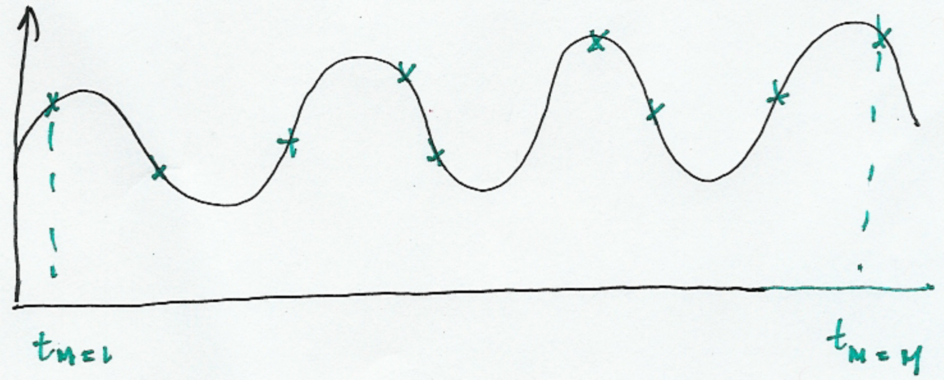
Example 1

$$y(t_n) = 10 \sin(2\pi s t_n) \qquad n = 1 \dots N, \quad t_n = n\Delta t$$

s = frequency

$$R(n) = \frac{10^2}{2} \cos(2\pi s\, n \Delta t)$$ Autocovariance function

suppose we sample at location $t_m \rightarrow y(t_m)$



and now we want to reconstruct the signal at t_n

$$y(t_n) = \sum_{m=1}^{M} \alpha_{nm}\, y(t_m)$$

once again

$y(t_n) \rightarrow \underline{y}$

$y(t_m) \rightarrow \underline{x}$

$$\hat{\underline{y}} = \underline{\underline{\alpha}}\, \underline{x}$$

$$\underline{\underline{C}}_{yx} = C(t_n, t_m) = R(n-m)$$

$$\underline{\underline{\alpha}} = \langle \underline{y}\, \underline{x}^T \rangle \langle \underline{x}\, \underline{x}^T \rangle^{-1} = \underline{\underline{C}}_{yx} C_{xx}^{-1} \qquad \underline{\underline{C}}_{xx} = \text{''} = \text{''}$$

(see figure) Example 2 $\rightarrow$ ENSO Time series