**Analysis of two or more time series**

Given \( x(t) \) and \( y(t) \) with normal distribution

\[ t_n = 1 \ldots N \text{ at } N = \text{length of series} \]

\[ \begin{align*}
\langle x \rangle &= \langle y \rangle = 0 \\
\langle x^2 \rangle &= \langle y^2 \rangle = 1
\end{align*} \]

Assume again that there is a correlation

\[ y(t_n) = \alpha x(t_n) + n(t_n) \]

\[ \alpha = \langle y x \rangle \]

\[ e^2 = \frac{\langle y x \rangle^2}{\langle y^2 \rangle} = \frac{\langle y x \rangle^2}{\langle y^2 \rangle} \]

**What is the significance of \( e^2 \)?**

To compute the significance we need to find

\[ P_{k_2, \text{d.o.f.}} \]

\[ R_H^2 \Theta = \sum_{i=1}^{M} \frac{(x_i y_i - \bar{x} \bar{y})^2}{\bar{y}^2} = (N-1) \frac{\langle xy \rangle^2}{\langle y^2 \rangle} \quad \text{d.o.f.} \]

\[ \frac{(N-1)\langle xy \rangle^2}{R^2 0.025, N} < e^2 < \frac{(N-1)\langle xy \rangle^2}{R^2 0.975, N} \]

Confidence at 95% level
Cross-Spectrum Analysis

Similar to the power spectrum, the cross-spectrum decomposes the correlation function between two signals as a function of the frequency.

Recall

\[ y(t) = x(t) y(t+\tau) + n(t) \]
\[ \alpha(t) = \frac{\langle y(t) y(t+\tau) \rangle}{\langle y(t)^2 \rangle} \]
\[ \hat{\alpha}(\tau) = \hat{x}(s) \hat{y}(s) \hat{s} = \hat{r}(s) = \int_{-\infty}^{\infty} \hat{x}(\tau) e^{-i\omega \tau} d\tau \]

Power spectrum of \( y(t) \)
\[ \hat{y}(s) \hat{y}^*(s) ds = \int y(t) y(t) dt \]

Parseval's Theorem

If we assume

\[ y(t) = x(t) \hat{x}(t+\tau) + n(t) \]
\[ \langle y \rangle = \langle nx \rangle = 0 \]
\[ \langle y^2 \rangle = \langle nx^2 \rangle = 1 \]

implies a correlation between \( x \) and \( y \)

\[ r_{xy}(\tau) = \langle y(t) x(t-\tau) \rangle \]
\[ r_{xy}(\tau) e^{-i\omega \tau} = \hat{r}(s) = \hat{x}(s) \hat{y}^*(s) = \hat{y}^*(s) \hat{x}(s) = \hat{f}(s) \]
\[ \hat{f}(s) = \hat{c}(s) + i \hat{q}(s) \]
\[ \Phi(s) = \cospectrum \quad \rightarrow \text{real part} \]

\[ \Psi(s) = \quad \text{quadrature spectrum} \quad \rightarrow \text{imaginary part} \]

**How to interpret the real and imaginary part of \( \Phi_y \)?**

\[ \Phi_y = \cospectrum(s) + i \Psi(s) = \hat{g}(s) \times^s(s) \]

\[ \cospectrum(s) e^{i \Psi(s)} e^{i \frac{\hat{g}(s)}{s}} \]

**Recall complex numbers**

\[ z = A e^{i \beta} = a + i b \]

```
<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
</tr>
</thead>
</table>
```

"complex plane"

\[ A e^{i \beta} = A \cos(\beta) + i A \sin(\beta) \]

```
a b
```

Phase difference
Recalling the complex plane

\[ Y(s) = C_y(s) e^{i \theta_y(s)} \]
\[ C_x(s) e^{-i \theta_x(s)} \]

1st vector in complex plane
2nd vector

Now you have to pieces of information

1. the amplitude of the covariability between x and y at each s

2. the phase relationship
e.g. if \( \theta_y = \theta_x \) \( \Rightarrow \) in phase
   
   \[ \text{if } \theta_y - \theta_x = \frac{\pi}{2} \]
   
   they are 90° out of phase with y leading x by \( \frac{\pi}{4} \)
However, the covariance function $r(t) = \langle y(t) \times (t-t') \rangle$ can be positive and negative. It is often more convenient to look at correlation

$$e^2(t) = \frac{\langle y(t) \times (t-t') \rangle^2}{\langle y(t)^2 \rangle \langle x(t)^2 \rangle}$$

correlation squared in spectral space.

$$\text{coh}^2(s) = \frac{\mathbb{E}[y^2]}{\mathbb{E}[y]^2} = \frac{\mathbb{C}(s)^2 + \mathbb{G}(s)^2}{\mathbb{G}(s)}$$

and like

$$\langle y^2 \rangle (1 - e^2) = \langle n^2 \rangle$$

physical space

$$\Phi_{yy}(1 - \text{coh}^2(s)) = \Phi_{nn}(s)$$

frequency space.

For each frequency it tells the amount of variance of $y$ that can be explained by $x$.

95% confidence limit. We deal with normal distribution with $n$ d.o.f. using the $\chi^2$ distribution.
Example

\[ y(t_n) = B_y \sin(2\pi s_0 t_n) \]
\[ x(t_n) = B_x \sin(2\pi s_0 t_n + \phi) \]

CASE 1

if \( \phi = 0 \) that \( y(t) \) and \( x(t) \) are identical and there covariance

\[ \langle y x \rangle = B_y \frac{B_x}{2} \]

CASE 2

However if \( \phi \neq 0 \) you can show that

\[ \langle y x \rangle = B_y B_x \left[ \cos(\phi) - i \sin(\phi) \right] \]

 Covariance in physical space is modulated by \( \cos \phi \)

"Fourier Transform"

\[ \hat{y}(s) = \int_{-\infty}^{\infty} y(t) e^{-i2\pi st} \, dt = \int_{-\infty}^{\infty} \left[ 0 + iB_y \right] e^{i2\pi s_0 t} e^{-i2\pi st} \, dt \]

\[ \delta(s-s_0) \]

\[ \hat{y}(s) = \frac{B_y}{\sqrt{2}} \left[ 0 + iB_y \right] \delta(s-s_0) \]

Similarly

\[ \hat{x}(s) = \frac{B_x}{\sqrt{2}} \left[ 0 + iB_x \right] \delta(s-s_0) e^{i\phi} \]
If we compute the covariance at the frequency $s = s_0$,

\[
\hat{g}(s_0) \hat{x}(s_0) = \frac{1}{\sqrt{2}} \left[ 0 + iB_y \right] \frac{1}{\sqrt{2}} \left[ 0 - iB_x \right] e^{-i\phi} = \Phi_{yx}
\]

\[
= \frac{1}{2} B_y B_x e^{-i\phi} = \frac{B_y B_x}{2} \left[ \cos \phi - i \sin \phi \right]
\]

where $\phi$ is the phase difference.

Covariance in physical space.

By $B_x \cos \phi = \cos$ spectrum

By $B_x \sin \phi = \text{quadrature}$ spectrum

\[
\text{coh}^2(s_0) = \frac{\Phi_{yy}}{\Phi_{yy} \Phi_{xx}}
\]

\[
\Phi_{yy}(s_0) = \frac{1}{2} B_y^2
\]

\[
\Phi_{xx}(s_0) = \frac{1}{2} B_x^2
\]

\[
\Phi_{yx}^2 = \frac{B_y^2 B_x^2}{4} \left[ \cos^2 \phi + \sin^2 \phi \right]
\]

\[
\text{coh}^2(s_0) = \frac{B_y^2 B_x^2}{4} \cdot \frac{4}{B_y^2 B_x^2} = 1
\]

At $s_0$ frequency all the variance of $y$ is explained by $x$.\]