Analysis of two or more time series

Given $x(t_i)$ and $y(t_i)$ with normal distribution

$t_n = 1 \ldots N\Delta t$ $\quad N$ = lenght of series $\quad \begin{cases} \langle x \rangle = \langle y \rangle = \emptyset \\ \langle x^2 \rangle = \langle y^2 \rangle = 1 \end{cases}$

Assume again that there is a correlation

$$y(t_n) = \alpha\, x(t_n) + n(t_n)$$

$$\alpha = \langle y x \rangle$$

$$e^2 = \frac{\langle y x \rangle^2}{\langle x^2 \rangle \langle y^2 \rangle} = \langle y x \rangle^2$$

What is the significance of e^2 ?

To compute the significance we need to find

$P_{R^2, d.o.f.}$

$$R^2_M = \sum_{i=1}^{M} \frac{(x_i y_i)^2}{e^2} = (N-1)\langle x y \rangle^2 \qquad M = d.o.f.$$

$$\frac{(N-1)\langle x y \rangle^2}{R^2_{0.975, M}} < e^2 < \frac{(N-1)\langle x y \rangle^2}{R^2_{0.025, M}}$$

confidence at 95% level

Cross-Spectrum Analysis

Similar to the powerspectrum, the cross-spectrum decomposes the ~~correlation function~~ covariance between two signals as a function of the frequency

Recall $\langle y \rangle = 0 \quad \langle y^2 \rangle = 1$

$$y(t) = \alpha(\tau)\, y(t-\tau) + n(t)$$

$$\alpha(\tau) = \frac{\langle y(t)\, y(t-\tau) \rangle}{\langle y(t)\, y(t) \rangle}$$ ← Autocovariance function $= r(\tau)$

Powerspectra of $y(t)$ $= \hat{y}(s)\, \hat{y}^*(s) = \hat{r}(s) = \int_{-\tau}^{+\tau} r(\tau) e^{-is\tau} d\tau$

Parseval Theorem

$$\underbrace{\int \hat{y}(s)\, \hat{y}^*(s)\, ds}_{\text{variance in frequency domain}} = \underbrace{\int y(t)\, y(t)\, dt}_{\text{variance in physical space}}$$

If we assume

$$y(t) = \alpha(\tau)\, x(t-\tau) + n(t) \qquad \begin{array}{l} \langle y \rangle = \langle x \rangle = 0 \\ \langle y^2 \rangle = \langle x^2 \rangle = 1 \end{array}$$

implies a correlation between x an y

$$r_{xy}(\tau) = \langle y(t)\, x(t-\tau) \rangle$$

$$\int_{-\tau}^{\tau} r_{xy}(\tau) e^{-is\tau} = \hat{r}(s) = \hat{y}(s)\, \hat{x}^*(s) = \Phi_{xy} = \underset{\text{real}}{CO(s)} + i\, \underset{\text{Imag.}}{Q(s)}$$

$co(s)$ = cospectrum → real part

$Q(s)$ = quadrature spectrum → immaginary part

How to interpret the real and immaginary part of Φ_{xy} ?

$$\Phi_{xy} = co(s) + i\,Q(s) = \hat{y}(s)\,\hat{x}^*(s)$$

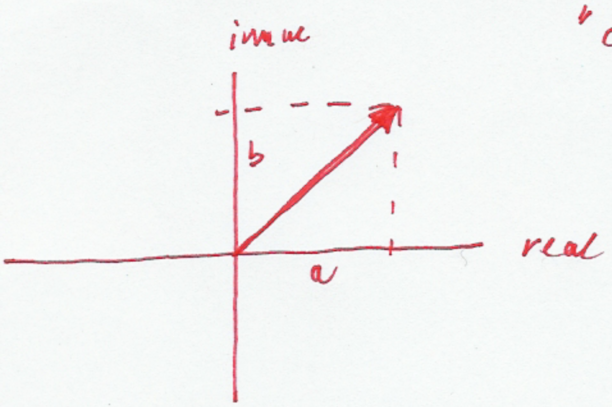
$$\hat{y}(s) = C_y(s)\,e^{i\vartheta_y(s)} \qquad \hat{x}^*(s) = C_x(s)\,e^{-i\vartheta_x(s)}$$

$$= \underbrace{C_y C_x(s)}_{\text{real}}\; e^{i\underbrace{(\vartheta_y(s) - \vartheta_x(s))}_{\text{phase difference}}}$$

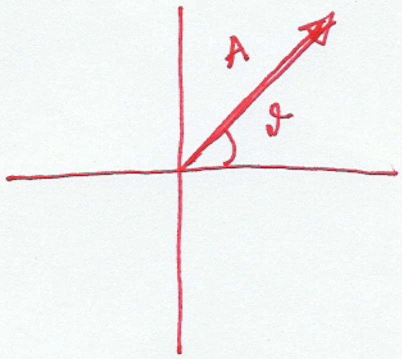
Recall complex numbers

$$z = A e^{i\vartheta} = a + ib$$

"complex plane"



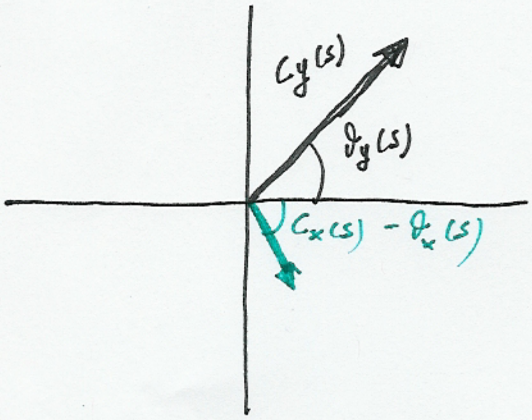
or

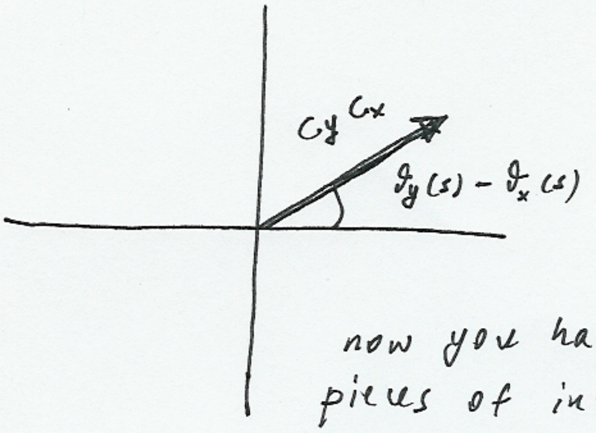


$$A e^{i\vartheta} = \underbrace{A\cos(\vartheta)}_{a} + \underbrace{iA\sin(\vartheta)}_{b}$$

Recaling the complex plan

$$\Phi_{xy}(s) = \underbrace{C_y(s)\, e^{i\vartheta_y(s)}}_{\text{1 vector in complex plane}} \; \underbrace{C_x(s)\, e^{-i\vartheta_x(s)}}_{\text{2nd vector}}$$





now you have to pieces of information

1. the amplitude of the covariability between x and y at each s

2. the phase relationship e.g. if $\vartheta_y = \vartheta_x$ → in phase

if $\vartheta_y - \vartheta_x = \frac{\pi}{2}$

they are 90° out of phase with y leading x by $\frac{\pi}{4}$

However the covariance function $r(\tau) = \langle y(t)\, x(t-\tau)\rangle$ can be positive and negative. It is often more convinient to look at correlation

$$\rho^2(\tau) = \frac{\langle y(t)\, x(t-\tau)\rangle^2}{\langle y(t)^2\rangle \langle x(t)^2\rangle}$$

coherence square in spectral space

$$\mathrm{coh}^2(s) = \frac{\Phi_{xy}^2}{\Phi_{yy}^{*}\,\Phi_{xx}^{*}} = \frac{CO(s)^2 + Q(s)^2}{\Phi_{yy}^{*}\,\Phi_{xx}^{*}}$$

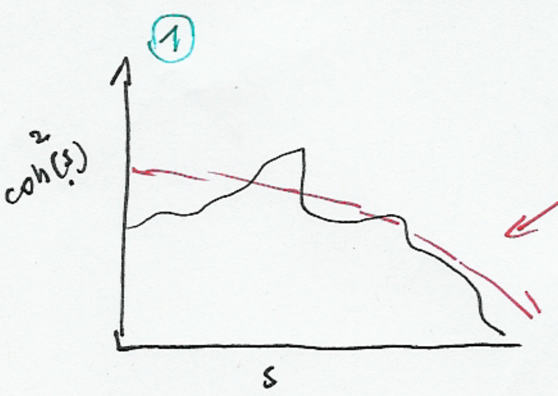
and like

$$\langle y^2\rangle (1 - \rho^2) = \langle n^2\rangle$$ physical space

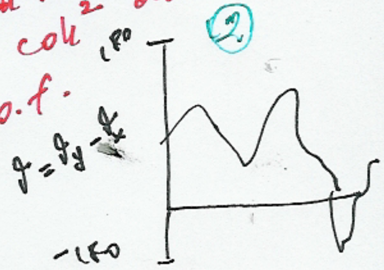
!

$$\boxed{\Phi_{yy}(s)\left(1 - \mathrm{coh}^2(s)\right) = \Phi_{nn}(s)}$$ frequency space.

for each frequency it tells the ammount of variance of y that can be explained by x



95% confidence limit. ~~You find this~~ ~~like we did with normal~~ ~~so~~ using the coh² distribution with n d.o.f.



Example

$$y(t_n) = B_y \sin(2\pi s_0 t_n)$$

$$x(t_n) = B_x \sin(2\pi s_0 t_n + \vartheta)$$

CASE 1

if $\boxed{\vartheta = 0}$ that $y(t)$ and $x(t)$ are identical and there covariance

$$\langle yx \rangle = \frac{B_y B_x}{2}$$

CASE 2

However if $\boxed{\vartheta \neq 0}$ you can show that

$$\langle yx \rangle = \underbrace{\frac{B_y B_x}{2} [\cos(\vartheta)}_{} \mp i \sin(\vartheta)]$$

covariance in physical space is modulated by $\cos\vartheta$

phase difference between x and y

" Fourier Transform "

$$\hat{y}(s) = \int_{-\infty}^{\infty} y(t) e^{-i2\pi st} dt = \int_{-\infty}^{\infty} [0 + iB_y] \underbrace{e^{i2\pi s_0 t} e^{-i2\pi st}}_{\delta(s-s_0)} dt$$

$$\hat{y}(s) = \frac{1}{\sqrt{2}} [0 + iB_y] \delta(s - s_0)$$

similarly $\longrightarrow$ $\hat{x}(s) = \frac{1}{\sqrt{2}} [0 + iB_x] \delta(s-s_0) e^{i\vartheta}$

if we compute the covariance at the frequency

$s = s_0$

~~<xy>~~ $$\hat{y}(s_0)\,\hat{x}^*(s_0) = \frac{1}{\sqrt{2}}[0 + iB_y]\frac{1}{\sqrt{2}}[0 - iB_x]\,e^{-i\vartheta} = \Phi_{yx}$$

$$= \frac{1}{2} B_y B_x\, e^{-i\vartheta} = \underbrace{\frac{B_y B_x}{2}}_{\text{covariance in physical space}}[\cos\vartheta - i\sin\vartheta]$$

phase difference

$$coh^2(s_0) = \frac{\Phi_{yx}^2}{\Phi_{yy}\,\Phi_{xx}}$$

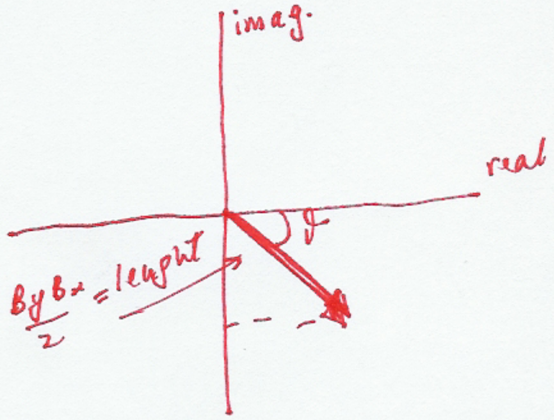
$\frac{B_y B_x}{2}\cos\vartheta$ = cospectrum

$\frac{B_y B_x}{2}\sin\vartheta$ = quadrature spectra

$$\Phi_{yy}(s_0) = \frac{1}{2} B_y^2$$

$$\Phi_{xx}(s_0) = \frac{1}{2} B_x^2$$

$$\Phi_{yx}^2 = \frac{B_y^2 B_x^2}{4}\,[\underbrace{\cos^2\vartheta + \sin^2\vartheta}_{1}]$$



$$coh^2(s_0) = \frac{B_y^2 B_x^2}{4} \cdot \frac{4}{B_y^2 B_x^2} = 1$$

At s_0 frequency all the variance of y is explained by x!