Time series Analysis

Discrete

We have seen how to extract the seasonal cycle from a time series $y(t)$

$$\hat{y}(t) = a\cos(\omega t) + b\sin(\omega t) + n(t)$$

$$\omega = \frac{2\pi}{12\ \text{months}}$$

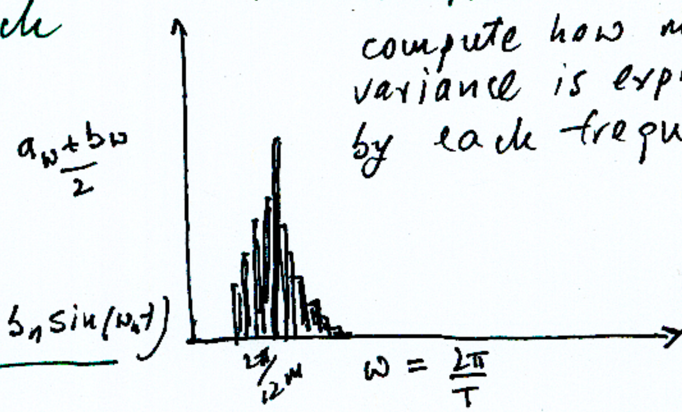
(using LSQ) $\rightarrow$ a, b the amplitude

$$\langle \hat{y}^2 \rangle = \frac{a^2+b^2}{2}$$ variance of the seasonal cycle.

Generalize for each frequency $\longrightarrow$

PERIODOGRAM

compute how much variance is explained by each frequency



Fourier series

$$\hat{y}(t) = \frac{1}{2}a_0 + \sum_{n=1}^{N} a_n\cos(\omega_n t) + b_n\sin(\omega_n t)$$

$$\omega_n = \frac{2\pi n}{T}$$ T = Time lenght of the $y(t)$

Continous form

$$y(t) = \int_{-\infty}^{\infty} a(\omega)\cos(\omega t) + b(\omega)\sin(\omega t)\,d\omega + \cancel{\frac{1}{2}a_0}$$

MEAN, let us assume for now that $\langle y \rangle = 0$

recall: $z = x + iy$

$$e^{i\vartheta} = \cos(\vartheta) + i\sin(\vartheta)$$

now set $\vartheta \equiv \omega t \qquad x \equiv a(\omega) \qquad y \equiv b(\omega)$

$$y(t) = \int_{-\infty}^{\infty} z(\omega)\, e^{-i\omega t}\, d\omega$$ ← the real part only

<u>proof:</u>

$$y(t) = \int_{-\infty}^{\infty} \left[a(\omega) + i\, b(\omega)\right]\left[\cos(\omega t) - i \sin(\omega t)\right] d\omega =$$

$$= \int_{-\infty}^{\infty} a(\omega)\cos(\omega t) - i\, a(\omega)\sin(\omega t) + i\, b(\omega)\cos(\omega t) \underbrace{- i^2}_{+1} b(\omega)\sin(\omega t)\, d\omega$$

$$= \int_{-\infty}^{\infty} \underbrace{a(\omega)\cos(\omega t) + b(\omega)\sin(\omega t)}_{\text{real part}} + i\underbrace{\left[b(\omega)\cos(\omega t) - a(\omega)\sin(\omega t)\right]}_{\text{immaginary part.}} d\omega$$

more general form of the continous

<u>FOURIER TRANSFORM</u>

$\omega = 2\pi s \qquad s$ = frequency

ω = radian frequency

$$y(t) = \int_{-\infty}^{\infty} \hat{y}(s)\, e^{-2\pi i s t}\, ds$$

if I want to derive $\hat{y}(s)$? (the complex amplitude of each frequency)

I apply the Fourier Transform $F(\,)$

$$F(y(t)) = \int_{-\infty}^{\infty} (y(t))\, e^{2\pi i s t}\, dt = \hat{y}(s)$$

$$\Rightarrow F^{-1}(\hat{y}(s)) = y(t)$$

proof:

$$\int_{-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} \hat{y}(s') e^{-2\pi i s' t} ds'}_{y(t)} \, e^{2\pi i s t} dt$$

$y(t)$ and the s' has been introduced to distinguish s' from s

$$= \int_{-\infty}^{\infty} \hat{y}(s') \int_{-\infty}^{\infty} e^{-2\pi i s' t} e^{2\pi i s t} dt \, ds'$$

$$= \int_{-\infty}^{\infty} \hat{y}(s') \underbrace{\int_{-\infty}^{\infty} e^{-2\pi i (s'-s) t} dt}_{\delta(s'-s)} \, ds' = \hat{y}(s)$$

Delta Function def:

$$\boxed{\begin{aligned} \delta(s'-s) &= \int_{-\infty}^{\infty} e^{-2\pi i (s'-s) t} dt \\ &= \int_{-\infty}^{\infty} e^{2\pi i (s-s') } dt \end{aligned}}$$

SUMMARY of FOURIER TRANSFORM PAIRS

$$F[y(t)] = \hat{y}(s) = \int_{-\infty}^{\infty} y(t) e^{\pm 2\pi i s t} dt$$

$$F^{-1}[\hat{y}(s)] = y(t) = \int_{-\infty}^{\infty} \hat{y}(s) e^{\mp 2\pi i s t} ds$$

signs are reversed you can choose $\pm$ or $\mp$ its the same.

Once we have $\hat{y}(s) = a(s) + i b(s)$ the variance of each frequency

$$\frac{a^2(s) + b^2(s)}{2} = \frac{\overbrace{\left[a(s) + i b(s)\right]}^{\hat{y}(s)}\overbrace{\left[a(s) - i b(s)\right]}^{\hat{y}^*(s)}}{2}$$

← complex conjugate

~~AF(f)~~ $\frac{\hat{y}(s)\,\hat{y}^*(s)}{2} = \frac{|\hat{y}(s)|^2}{2}$ is called POWER SPECTRUM of $y(t)$

= how the variance of the signal is distributed among the various frequencies.

PARSEVAL'S THEOREM

$$\int_{-\infty}^{\infty} y(t)^2\, dt = \int_{-\infty}^{\infty} |\hat{y}(s)|^2\, ds$$

(note: when computing variance you may divide by the length of your sample in the discrete sample)

Variance in physical space = sum of the variances at each frequency

$(a+ib)(\cos(t) - i\sin t) \longrightarrow$ same as $\longrightarrow (a-ib)(\cos t + i \sin t)$

proof, ① $y(t) = \int_{-\infty}^{\infty} \hat{y}(s)\, e^{-i2\pi s t}\, ds$ and also true

② $y(t) = \int_{-\infty}^{\infty} \hat{y}^*(s')\, e^{+i2\pi s' t}\, ds'$

$\boxed{\text{note: implies } \hat{y}^*(s') = \hat{y}(-s)}$

$$\int_{-\infty}^{\infty} y(t)^2\, dt = \iiint_{-\infty}^{\infty} \hat{y}(s)\, e^{-i2\pi s t}\, \hat{y}^*(s')\, e^{+i2\pi s' t}\, ds\, ds'\, dt$$

$$= \iint_{-\infty}^{\infty} \hat{y}(s)\, \hat{y}^*(s') \underbrace{\int_{-\infty}^{\infty} e^{-i2\pi(s-s')t}\, dt}_{\delta(s-s')}\, ds\, ds' = \int_{-\infty}^{\infty} \underbrace{\hat{y}(s)\hat{y}^*(s)}_{|\hat{y}(s)|^2}\, ds$$

Often we like to compute the so called autocovariance of a signal $y(t)$ and build a model like

$$y(t+1) = \alpha\, y(t) + n(t)$$

$$\alpha = \frac{\langle y(t+1)\, y(t) \rangle}{\langle y(t)\, y(t) \rangle}$$

autocovariance between t and $t+1$

we can expand this model to be general and

$$y(t+t') = \alpha(t+t')\, y(t) + n(t)$$

Time interval

$$\alpha(t') = \frac{\langle y(t+t')\, y(t) \rangle}{\langle y(t)\, y(t) \rangle}$$

autocovariance of the function of the Δt interval $\Delta t \equiv t'$ $h(t')$

now alfa is a continous function.

Let us assume that $t \in [-\infty \ \infty]$

$$\langle y(t+t')\, y(t) \rangle \approx \int_{-\infty}^{\infty} y(t+t')\, y(t)\, dt = h(t')$$

if we look at this in fourier space $\hat{h}(s)$

$$\boxed{\hat{h}(s) = \hat{y}^*(s)\, \hat{y}(s)}$$

same as POWER SPECTRA

(1) which says that the autocovariance function has power for lags in time $t' \equiv \Delta t$ for which there is a peak in the spectra at $1/t' = s$

Convolution Theorem

Autocovariance function + ~~5t~~

$$h(t) = \int_{-\infty}^{\infty} f(t')\, g(t \pm t')\, dt'$$

$$= \int_{-\infty}^{\infty} f(t') \int_{-\infty}^{\infty} \hat{g}(s)\, e^{i 2\pi s (t \pm t')}\, ds\, dt'$$

$$= \int_{-\infty}^{\infty} \hat{g}(s) \underbrace{\int_{-\infty}^{\infty} f(t')\, e^{\pm i 2\pi s t'}\, dt'}_{\hat{f}^{*}(s)}\, e^{i 2\pi s t}\, ds$$

$$= \int_{-\infty}^{\infty} \underbrace{\hat{g}(s)\, \hat{f}^{*}(s)}_{\hat{h}(s)}\, e^{i 2\pi s t}\, ds$$

Assume I wanted to compute a more fancy model

$$y(t+1) = \alpha_1 y(t) + \alpha_2 y(t-1) + \alpha_3 y(t-2) \dots$$

$$y(t) = \int_{-\infty}^{+\infty} \alpha(t')\, y(t-t')\, dt'$$

$$\alpha(t') \to \frac{h(t')}{\langle y(t)^2 \rangle}$$

Assume that

$$y(t) = \frac{1}{\langle y(t)^2 \rangle} \int_{-\infty}^{\infty} h(t')\, y(t-t')\, dt'$$

Assume $h(t') = \int_{-\infty}^{\infty} \hat{h}(w)\, e^{i2\pi st}\, ds$

for simplicity assume there is only ~~one~~ 2 dominant frequency s_1 and s_2

$$\boxed{h(t') = \int h(s_1)\, e^{i2\pi s_1 t'} + h(s_2)\, e^{i2\pi s_2 t'}}$$

$$y(t) = \frac{1}{\langle y(t)^2 \rangle} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s_1)\, \hat{y}(s)\, \underbrace{e^{+i2\pi s_1 t' - i2\pi s t'}}_{\delta(s_1 - s)}\, e^{i2\pi s t}\, dt'\, ds$$

$$dt'\, ds$$

$$+ \iint h(s_2) \dots$$

$$y(t) = \hat{h}(s_1)\, \hat{y}(s_1)\, e^{i2\pi s_1 t} + \hat{h}(s_2)\, \hat{y}(s_2)\, e^{i2\pi s_2 t}$$

The real part of that is:

$$y(t) = \frac{1}{\langle y(t)^2\rangle} Re\left(\hat{h}(s_1)\,\hat{y}(s_1)\, e^{i2\pi s_1 t} + \dots \right)$$

$$\approx \frac{1}{\langle y(t)^2\rangle}\left[a(s_1)\underset{\sin}{\cos}(2\pi s_1 t) + a(s_2)\underset{\sin}{\cos}(2\pi s_2 t)\right]$$

$$a(s_1) = Real\left\{ \hat{h}(s_1)\,\hat{y}(s_1) \right\}$$

$$a(s_2) = Real\left\{ \hat{h}(s_2)\,\hat{y}(s_2) \right\}$$

Let us go back to

$$y(t) = \int_{-\infty}^{\infty} \alpha(t')\, y(t-t')\, dt'$$

This is a convolution integral.

These integral are handy and often used to smooth a signal. (e.g. we assume that the autocovariance function $\alpha(t')$ had only two frequency which lead to a signal with only two carrying frequency.)

Now let us assume that

$\cancel{\alpha(t') = H(t>t')}$

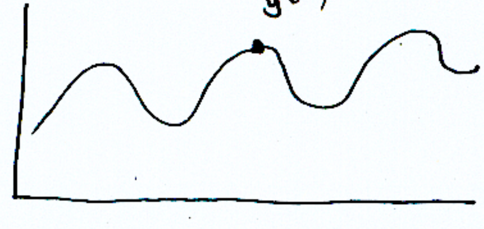
$$\hat{y}(t) = \int_{-\infty}^{\infty} H(t')\, \overset{-t'}{y(t')}\, dt' + \int_{-\infty}^{\infty} H(-t')\, y(t+t')$$

Heavyside Function

$H(t-t') = 1 \quad t>t'$

$ = 0 \quad t<t'$

$$H(s) = \int_{-\infty}^{\infty} e^{-2\pi i s t}\, dt = \frac{1}{2}\left[\delta(s) - \frac{i}{\pi s}\right] y(t)$$



$$\hat{y}(t) = \int_{-\infty}^{\infty}$$