Reference Chap. 2 of
Discrete Inverse ... book.
pages. 18-27

Vectors

$$\underline{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_N \end{bmatrix}$$

length of a vector

$$\sqrt{\sum_{i=1}^{N} q_i^2} = \sqrt{\underline{q}^T \underline{q}}$$

$$\underline{q}_{N\times 1}\, \underline{q}^T_{1\times N} = \underline{\underline{Q}}_{NN} \quad \text{(Matrix)}$$

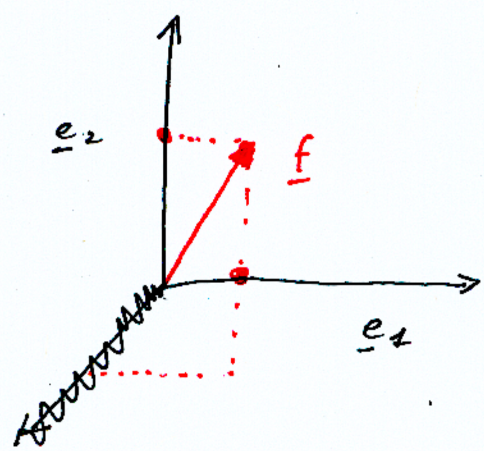
$$\underline{q}^T_{1\times N}\, \underline{q}_{N\times 1} = \text{scalar}$$

suppose we have a collection of ~~3~~ 2 vectors $\underline{e}_i$

$$\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \cancel{0} \end{bmatrix} \qquad \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \cancel{0} \end{bmatrix} \qquad \cancel{\underline{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

assume these ~~three~~ two vectors is the "cartesian coordinates" in ~~3~~ 2 dimensions

Any vector $\underline{f}$ in the 2D cartesian framework can be represented as a linear sum of $\underline{e}_i$



$$\underline{f} = \sum_{i=1}^{2} \alpha_i \underline{e}_i$$

in this case $\underline{e}_i$ are said to be a "basis set"

A basis set $\underline{e}_i$ has the property that no $\underline{e}_i$ can be represented by the others e.g. 2

$$\underline{e}_1 \neq \alpha_2 \underline{e}_2$$ or more generally

$$\underline{e}_j \neq \sum_{i=1}^{N} \underline{e}_i \Longrightarrow \underline{e}_i^T \underline{e}_j = \delta_{ij}$$ orthogonal basis set

e.g.

$$\underline{e}_1^T \underline{e}_2 = [1 \ 0] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \phi + \phi$$

$$\underline{e}_1^T \underline{e}_1 = [1 \ 0] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 + \phi = 1$$

To find the coefficients α_i you compute the dot product with each $\underline{e}_i$

$$\underline{e}_i^T \underline{f} = \underline{e}_i^T \sum_{j=1}^{N} \alpha_j \underline{e}_j = \sum_{j=1}^{N} \underbrace{\underline{e}_i^T \underline{e}_j}_{\delta_{ij}} \alpha_j = \alpha_i$$

Assume $\underline{f} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$$\underline{e}_1^T \underline{f} = [1 \ 0] \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 = \alpha_1$$ projection on x-axis.

You can have basis set in multidimensions N

e.g. Fourier series

$$\underline{f} = \begin{bmatrix} f_1(t=0) \\ \vdots \\ \vdots \\ f_M(t=2\pi) \end{bmatrix} \qquad f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{N} a_n \cos(nt) + \sum_{n=1}^{N} b_n \sin(nt)$$

$$\underline{f} = \sum_{i=1}^{2N+1} \alpha_i \cdot \underline{e}_i \qquad \underline{e}_1 = \begin{bmatrix} 1/2 \\ \vdots \\ \vdots \\ 1/2 \end{bmatrix} \qquad \underline{e}_2 = \begin{bmatrix} \cos(1 \cdot t=0) \\ \vdots \\ \vdots \\ \cos(1 \cdot t=2\pi) \end{bmatrix}$$

Multidimensional set where the sin/cos are the basis set also called the "Fourier series" or "Fourier expansion"...

$$\underline{e}_3 = \begin{bmatrix} \cos(2 \cdot t=0) \\ \vdots \\ \vdots \\ \cos(2 \cdot t=2\pi) \end{bmatrix} \quad \cdots \quad \underline{e}_i = \begin{bmatrix} \sin(t=0) \\ \vdots \\ \vdots \\ \sin(t=2\pi) \end{bmatrix}$$

I can also write more concisely

$$\underline{f} = \underline{\underline{E}}\, \underline{\alpha} \qquad \underline{\underline{E}} = \begin{bmatrix} \underline{e}_1 & \underline{e}_2 & \cdots & \underline{e}_M \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$\underline{f}_{M\times 1} = \underline{\underline{E}}_{M\times(2N+1)}\, \underline{\alpha}_{(2N+1)\times 1}$$

~~Very~~ Verify that $\underline{e}_i^T \underline{e}_j = \delta_{ij}$ ✓

~~Matrix Identities~~

A_{MN} — M rows, N column

$$\underline{\underline{A}} = \begin{bmatrix} \vdots & \vdots & \vdots \end{bmatrix} \qquad \underline{a} = \begin{bmatrix} \vdots \end{bmatrix}$$

$$\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{C}} \neq \underline{\underline{B}}\,\underline{\underline{A}}$$

$$\underline{\underline{A}}^2 = \underline{\underline{A}}\,\underline{\underline{A}}$$

if $\underline{\underline{A}}$ is symmetric

$$\underline{\underline{A}} = \underline{\underline{A}}^T \qquad \underline{\underline{A}}^2 = \underline{\underline{A}}^T\underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{A}}^T$$

$$(\underline{\underline{A}}\,\underline{\underline{B}})^T = \cancel{A}\, \underline{\underline{B}}^T \underline{\underline{A}}^T$$

$$\underline{\underline{I}} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix} \qquad \text{identity matrix}$$

$\underline{\underline{E}}\,\underline{\underline{B}} = \underline{\underline{I}}$ then $\cancel{A}$ $\underline{\underline{B}}$ is the inverse of $\underline{\underline{E}}$
also denote $\underline{\underline{E}}^{-1}$

$$(\underline{\underline{A}}\,\underline{\underline{B}})^{-1} = \underline{\underline{B}}^{-1}\underline{\underline{A}}^{-1}$$

define

$$J = \underline{x}^T \underline{\underline{A}}\, \underline{x} \quad \text{is positive}$$

$\underline{\underline{A}}$ is said to be "positive semi-definite" or "non-negative" if $J \geq 0$ for any $\underline{x}$

If $\underline{\underline{A}}$ is positive definite symmetric

$$\underline{\underline{A}} = \underline{\underline{A}}^{T/2}\, \underline{\underline{A}}^{1/2}$$

Differentiation

$$\frac{\partial s}{\partial \underline{q}} = \begin{bmatrix} \partial s / \partial q_1 \\ \vdots \\ \partial s / \partial q_M \end{bmatrix}$$

$$\begin{array}{cc} & N \\ M & \frac{\partial \underline{r}}{\partial \underline{p}} = \begin{bmatrix} \frac{\partial r_1}{\partial p_1} & \cdots & \frac{\partial r_N}{\partial p_1} \\ \frac{\partial r_2}{\partial p_2} & & \vdots \\ & & \vdots \\ \frac{\partial r_1}{\partial p_M} & & \frac{\partial r_N}{\partial p_M} \end{bmatrix} \equiv \underline{\underline{B}} \quad M \times N \end{array}$$

The Jacobian of $\underline{r}$ is $\det[\underline{\underline{B}}]$

$\frac{\partial^2 s}{\partial \underline{q}^2}$ = is a matrix referred as the Hessian.