# Statistics of Joint variable $P_{xy}$

**Content:**
- LSQ
- JPDF estimates
- CHI-square

1. Assume linear $\boxed{y = \alpha x + n}$
2. Assume $P_{xy}$ is a joint Gaussian

$$P_{xy} = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-e^2}} \exp\left[-\frac{1}{2\sqrt{1-e^2}}\left(\frac{(x-\mu_x)^2}{\sigma_x^2} - \frac{2e(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \frac{(y-\mu_y)}{\sigma_y^2}\right)\right]$$

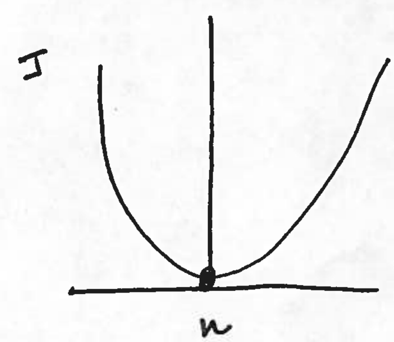
$$e = \frac{\langle xy \rangle}{\sigma_x\sigma_y} = \text{correlation}$$

- **Estimating $\alpha$ using Least square (LSQ)**

From 1 we can solve for $\alpha$ using LSQ

a) Define cost function $J = \langle n^2 \rangle$

b) Find Minimum of $J$




$$J = \langle (y - \alpha x)^2 \rangle$$

$$\frac{\partial J}{\partial \alpha} = 0 = \langle 2(y - \alpha x)(-x) \rangle$$

$$= 2\langle(-yx + \alpha x x)\rangle$$

$$= \underbrace{2\left[-\langle yx \rangle + \alpha \langle x x \rangle\right]}_{=0} \rightarrow \alpha = \frac{\langle xy \rangle}{\sigma_x^2}$$

covariance

variance

if we compare $e$ and $\alpha$ (2)

$$\boxed{e = \alpha \frac{\sigma_x}{\sigma_y}}$$

• The $\alpha$ from LSQ maximized JPDF conditional $P_{y|x=X}$

let us assume (2) with $\mu_x = \mu_y = \emptyset$ zero mean

$\sigma_x^2 = \sigma_y^2 = 1$ unit variance

$$P_{xy} = \frac{1}{2\pi\sqrt{1-e^2}} \exp\left[-\frac{1}{2\pi\sqrt{1-e^2}} \underbrace{\left(x^2 - 2exy + y^2\right)}_{ARG}\right]$$

If we want to find the most likely value of $y$ for a given $x$?

$$\frac{P_{y|x=X}}{P_{yx}} = P_y \longrightarrow \int_{-\infty}^{\infty} y\, P_y\, dY = \hat{y}$$

this is equivalent of MAX $\frac{P_{y|x=X}}{P_x}$

that is finding ARG = Maxima when $x = X$

$$\text{arg} = X^2 - 2eXy + y^2$$

$\alpha$

same as LSQ

$$\frac{\partial\, \text{arg}}{\partial y} = -2\alpha X + 2y = 0 \longrightarrow \hat{y} = \alpha X$$

$\boxed{\text{MAX } P_{xy} = \text{MIN } J}$ ✓

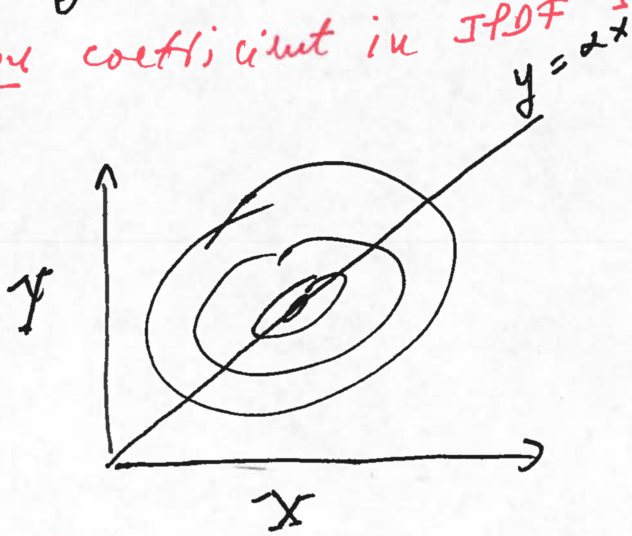
This estimate is also the one inferred from **LINEAR OPTIMAL ESTIMATORS**

$$y = \alpha x + n$$

$$\langle yx\rangle = \alpha \langle xx\rangle + \underset{=0}{\cancel{\langle xn\rangle}}$$

$$\alpha = \frac{\langle yx\rangle}{\langle xx\rangle}$$ regression coefficient in JPDF space

covariance between x and y




- statistics of cost function J → follow CHI-SQUARE

How to test the validity of this assumption? (e.g. linear relationship)

if $P_{xy}$ is gaussian → $P_n$ = Gaussian

LSQ $J = \langle n^2\rangle$ $\quad J^* = n_1^2 + n_2^2 + n_3^2 + \dots + n_K^2$

$$J^* = \sum_{k=1}^{K} \frac{n_k^2}{\sigma_n^2}$$ → follows CHI distribution with K d.o.f.

$\sigma_n^2$ = error variance $\quad \chi^2(K) = \frac{1}{2^{K/2}\Gamma(K/2)} x^{K/2} e^{-x/2}$